Winding angle variance of Fortuin-Kasteleyn contours

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The variance in the winding number of various random fractal curves, including the self-avoiding walk, the loop-erased random walk, contours of Fortuin-Kastelyn clusters, and stochastic Loewner evolution, has been studied by numerous researchers. Usually the focus has been on the winding at the end points. We measure the variance in winding number at typical points along the curve. More generally, we study the winding at points where k strands come together, and some adjacent strands may be conditioned not to hit each other. The measured values are consistent with an interesting conjecture.

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Duplantier and Saleur [1] studied the winding angle between the two endpoints of a finite self-avoiding walk (SAW) in two dimensions (2D), and indeed, a broader class of curves. Using exact but nonrigorous Coulomb gas methods, they found that the distribution of winding angle approaches a Gaussian and they explicitly computed the variance. When the end points of the walk are distance L apart, the winding variance is $\sim(8/g)\ln L$, where g is a model-dependent parameter which is 3/2 for SAW. The winding angle at a single end point (relative to the global average direction of the curve—see below for a precise definition) is a Gaussian with variance $(4/g)\ln L$ [1]. We found experimentally that the variance in the winding at typical (random) points along the curve was only 1/4 as large as the variance in the winding at the end points. More generally, when k strands of the curve come together at a point, the winding angle variance is $1/k^2$ as large as at the end points; Eq. (1) below generalizes this further.

Remark. After our initial experiments we learned that the $4/(gk^2)\ln L$ formula is also contained in unpublished notes of Duplantier. However, to our knowledge, the winding at typical points or points where k strands come together is not mentioned in the literature, except in the case of loop-erased random walk (LERW), where we have identified a minor oversight in the calculations that resulted in incorrect values being reported. Our experiments can be seen as a test of Duplantier's Coulomb gas predictions. We also report on other random fractal curves, for which the Coulomb gas methods do not apply.

The main object of our study is the 2D Fortuin-Kasteleyn (FK) random cluster model [2,3] at criticality, specifically the contours of the clusters. The FK model is like bond percolation with edge probability p, but there is another parameter q, and $\Pr[\text{configuration}]\propto p^{\#\text{'s bonds}}(1-p)^{\#\text{'s missing bonds}}q^{\#\text{'s clusters}}$. For each q, there is a critical p above which the system percolates. When $p=p_{\text{critical}}$, the contours of the clusters form a system of loops called the fully packed loop model [4–6]. See Fig. 1, left panels. A loop configuration occurs with probability proportional to $n^{\#\text{'s loops}}$ where $n=\sqrt{q}$ [4–6].

In addition to the perimeter (or "hull") of a cluster, the "external perimeter" has also been studied (see Fig. 1, right panels). Grossman and Aharony [7] found experimentally that by closing off narrow passageways on the hull of percolation clusters, the fractal dimension drops from 7/4 [8] to 4/3, and that furthermore the precise definition of "narrow" had little or no effect on the fractal dimension 4/3. See Ref. [9] for an explanation of this phenomenon in terms of path crossing exponents. The external perimeter of percolation clusters is believed to be essentially the self-avoiding walk [7,9–11], and the external perimeters of FK clusters for other values of q are also interesting. Therefore, in addition to studying the hulls of the FK clusters, we also studied their external perimeters by closing off passageways of lattice spacing 1 and looking at the hulls of the resulting clusters.

The perimeters and external perimeters of FK clusters are

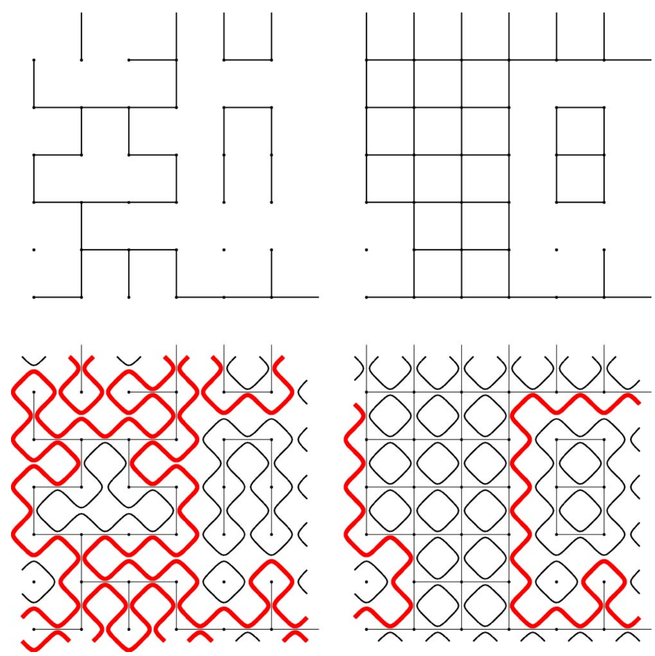

FIG. 1. The contours and external perimeter of critical FK clusters. The upper left panel shows a FK random cluster configuration with $q=3$ at criticality. On the upper right panel we closed off the narrow passageways of this FK configuration by connecting adjacent pairs of vertices that belong to the same connected component. In the lower panels we show the fully packed loop configurations that come from the above bond configurations. The loops on the left traverse the hull or perimeter of the clusters. The loops on the right traverse the external perimeter of the clusters. The longest perimeter loop and longest external perimeter loop are shown in bold.

TABLE I. Summary of κ_k measurements for the perimeter and external perimeter of FK clusters. The table goes up to $q=4$, beyond which the FK model has a first-order phase transition without large loops [16,17]. The values of the fractal dimension D_f are summarized from Refs. [7–9,14,18–23], and κ_1 comes from Ref. [1]. The values for κ_2 for the perimeter and external perimeter were measured as described below. κ_3 for LERW, was measured by looking at the "triple points" of uniformly random spanning trees—points where three strands come together. κ_4 and κ_6 for the spanning tree perimeter are κ_2 and κ_3 for LERW, respectively. For $q=1$, κ_3 is κ_2 of the external perimeter [9,24], and likewise $\kappa_5=\kappa_4'$ (defined below). The measurements are consistent with the hypothesis that $\kappa_k=\kappa_1/k^2$, a formula that also appears in [25].

			Perimeter of FK clusters								External perimeter of FK clusters						
q [a]	n [b]	g [c]	κ_1 [d]	κ_2	κ_3	κ_4	κ_5	κ_6	D_f [e]	Related model	q	g	κ_1	κ_2	κ_3	D_f	Related model
0	0	1/2	8	2		1/2		2/9	2	Spanning tree dense SAW	0	2	2	1/2	2/9	5/4	LERW
1	1	2/3	6	3/2	2/3		6/25		7/4	Polymers at Φ point square ice	1	3/2	8/3	2/3		4/3	Self-avoiding walk Brownian frontier
2	$\sqrt{2}$	3/4	16/3	4/3					5/3		2	4/3	3	3/4		11/8	
$\frac{3+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	4/5	5	5/4					13/8		$\frac{3+\sqrt{5}}{2}$	5/4	16/5			7/5	
3	$\sqrt{3}$	5/6	24/5	6/5					8/5		3	6/5	10/3			17/12	
4	2	1	4	1					3/2	Two perfect matchings	4	1	4			3/2	

[a] q is the cluster fugacity in FK random cluster model. $n^2=q$ [4–6] (perimeter only, not external perimeter).
[b] n is the loop fugacity in fully packed loop model. $n=-2\cos(\pi g)$ [12,13].
[c] g is the coupling constant of the associated Coulomb gas. $\kappa=\kappa_1=4/g$ [1]; $g_{\text{ext. perimeter}}=1/g_{\text{perimeter}}$ [14].
[d] κ_k is the winding angle variance when k curves meet at a point.
[e] D_f is the fractal dimension of curves. $D_f=1+1/(2g)=1+\kappa/8$ [12,15].

closely related to a variety of models, most notably the stochastic Loewner evolution (SLE) process introduced by Schramm [26]. In a discretized version of SLE, a curve in the plane grows as follows: the portion of the plane not in the curve is conformally mapped to the half plane, with the tip of the curve mapped to the origin. A small random cut is then made starting at the origin, where the slope of the cut is controlled by a parameter κ. The original curve gets extended by the preimage of this small cut, and the process repeats. The variance in the winding angle at the end point of SLE_κ is $\kappa \ln L$ [26].

The SLE process describes the limiting behavior of a variety of statistical mechanical models in 2D. Schramm proved that SLE_2 gives the scaling limit of LERW, provided that LERW has a conformally invariant scaling limit; recently Lawler, Schramm, and Werner [27] proved this without assumptions. Smirnov [28] proved that critical site percolation in the triangular lattice converges to SLE_6. Lawler, Schramm, and Werner [29] proved that the frontier of Brownian motion (with suitable boundary conditions) converges to $\mathrm{SLE}_{8/3}$. There are good theoretical [30] and experimental [31] reasons to believe that $\mathrm{SLE}_{8/3}$ also describes the SAW. Calculations by Kenyon and Schramm [32] suggest that SLE_4 describes the loops arising from superimposing two domino tilings. Rohde and Schramm [15] proved that the fractal dimension D_f of SLE_κ is at most $1+\kappa/8$ when $\kappa \leqslant 8$, and their calculations suggest $D_f=1+\kappa/8$. See also Refs. [18,20,33–37] for further results on SLE. Schramm conjectured that the contours of FK clusters at criticality have the same local properties as SLE_κ, where κ depends on q.

We study the perimeter and external perimeter of FK clusters by looking at the winding angle function. Given a loop or a path in the plane or on a torus, we define the winding angle function $w()$, a function of the edges, as follows. We pick an arbitrary starting edge e on the loop or path and an

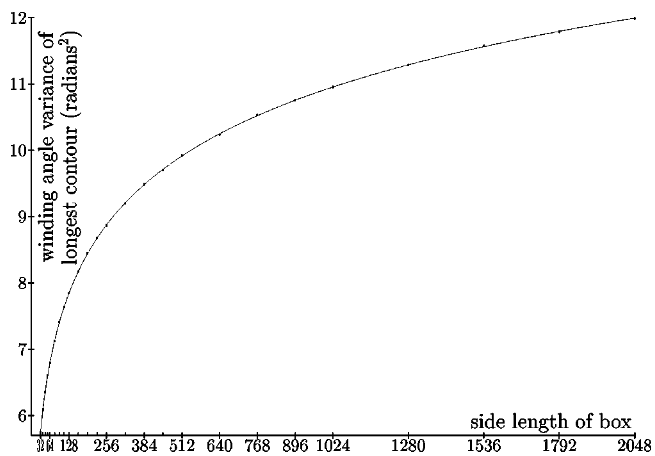

FIG. 2. Winding angle variance for the largest contour when $q=1$ as a function of the side length L of the box. Error bars on our estimates of the winding angle variance are shown, but are quite short and appear as points. A curve of the form $\kappa_2 \ln L + a$ was least-squares fitted to these data and plotted here. The 95% confidence intervals (± 1.96 standard deviations) for the parameters are $\kappa_2=1.5002\pm 0.0023$ and $a=0.554\,94\pm 0.013$, consistent with $\kappa_2=3/2$. The fit has a χ^2 statistic of 23.65 with 23 degrees of freedom for a p value of 0.42, so the fit passes the χ^2 test. In this case the fit is good all the way down to $L_{\min}=32$. In some cases the fitted curve lies outside the 95% confidence interval at $L_{\min}$, indicating corrections to scaling, and in these cases we increased $L_{\min}$. These data are summarized in the first line of Table II.

TABLE II. Winding angle variance coefficient for the longest loop (κ_2), longest loop of the external perimeter (κ_2'), and largest pinch of the longest loop (κ_4'). When $q=4$, we expect that log corrections [43,44] affect the measured κ_2' and κ_4'.

q	κ_2	Nearby rational	L's	χ^2 test p value
1	1.500 ± 0.002	3/2	32–2048	0.42
2	1.333 ± 0.003	4/3	32–1280	0.77
$\frac{3+\sqrt{5}}{2}$	1.252 ± 0.003	5/4	32–896	0.036
3	1.204 ± 0.004	6/5	32–896	0.59
4	1.078 ± 0.007	1?	32–768	0.72
q	κ_2'	Nearby rational	L's	χ^2 test p value
1	0.666 ± 0.002	2/3	80–2048	0.71
2	0.747 ± 0.002	3/4	32–1280	0.67
$\frac{3+\sqrt{5}}{2}$	0.779 ± 0.003	4/5?	32–896	0.66
3	0.795 ± 0.005	??	32–896	0.019
4	0.800 ± 0.008	??	32–768	0.75
q	κ_4'	Nearby rational	L's	χ^2 test p value
1	0.239 ± 0.007	6/25	32–2048	0.84
2	0.247 ± 0.008	12/49?	32–1280	0.27
$\frac{3+\sqrt{5}}{2}$	0.243 ± 0.009	20/81?	32–896	0.31
3	0.243 ± 0.009	30/121?	32–896	0.71
4	0.267 ± 0.010	1/4??	32–768	0.86

arbitrary value for the winding function $w(e)$ at that edge. The winding at a neighboring edge e' is defined by $w(e') = w(e)+$ the turning angle from e to e' measured in radians. This definition applies to paths or noncontractable loops, i.e., loops that wind around the torus. If a loop is contractable to a point, this would yield a multivalued winding function, so we adjust the definition of $w(e')-w(e)$ by $2\pi/(\text{length of loop})$ to get a single-valued winding function. This specifies the winding angle function up to a global additive constant; we choose the value of this global constant to make the average winding angle of the edges on the loop or path 0.

When k strands of the perimeter or external perimeter converge on a point, the winding angle variance should scale as $\kappa_k \ln L$. Table I summarizes our measurements of κ_k, suggesting $\kappa_k = \kappa_1/k^2$ (see also Refs. [25,38]).

Remarks on LERW. Our simulation values for κ_2 and κ_3 for LERW disagree with the values previously reported by Kenyon [39] by a factor of 4 and 9, respectively. These calculations used the Temperley [40] correspondence between spanning trees and dimer systems, and Kenyon correctly and rigorously computed the variance in the height function of the associated dimer system when there were 1, 2, or 3 paths approaching a point. The height function of the dimer system is related to the winding angle for the paths: when there are

TABLE III. Measurements of κ_1, κ_2, κ_3, and D_f for the paths in uniform spanning tree (UST) (i.e., LERW [45]) and the minimum spanning tree (MST). The estimate of κ_1 comes from κ_2 of the spanning tree contour. The first estimate of κ_3 comes from a triple point, the second estimate comes from the longest pinch (κ_4') of the spanning tree contour.

UST path (LERW)	Parameter	Nearby rational	L's	χ^2 test p value
κ_1	2.000 ± 0.002	2	32–1792	0.024
κ_2	0.510 ± 0.003	1/2	48–1792	0.60
κ_3	0.235 ± 0.010	2/9	40–1792	0.17
κ_3	0.229 ± 0.008	2/9	32–1792	0.34
D_f	1.252 ± 0.001	5/4	32–1792	0.83
MST path				
κ_1	1.886 ± 0.001	?	32–4096	0.81
κ_2	0.439 ± 0.002	?	40–4096	0.0051
κ_3	0.200 ± 0.006	?	32–4096	0.97
κ_3	0.201 ± 0.006	?	32–4096	0.53
D_f	1.218 ± 0.001	?	32–4096	0.070

k paths, each winding changes the dimer height function by $4k$. The factor of k was omitted, leading to the factor of k^2 discrepancy in the winding angle variance.

The longest contour of a FK configuration is likely to hit itself many times (which is why the perimeter and external perimeter are different); the places where the contour hits itself are called pinch points. For the longest contour, we identified the pinch point giving rise to the longest pinch. At this point there are four strands that travel a distance of the order of the box length L, suggesting that the winding angle variance at this point should grow as $\kappa_4 \ln L$. However, as noted by Schramm [24], the pinch point with the longest pinch is an atypical pinch point because there are two adjacent strands conditioned not to hit each other—if they did hit each other, then this would create a longer pinch. Thus, the winding angle variance at the longest pinch point is governed by a different constant κ_4', and grows as $\kappa_4' \ln L$.

In general, let κ_k' be the winding angle variance coefficient when there are k strands meeting at a point and two adjacent strands do not hit each other (when $k=2$, the left side of one strand may hit the right side of the other strand, but not vice versa). When $q=4$, the strands do not hit each other anyway [15], so $\kappa_k' = \kappa_k$. When $q=1$ and k strands meet at a point, conditioning two adjacent strands not to touch has the same effect as adding an extra strand: $\kappa_k' = \kappa_{k+1}$ [9,24]. For other values of q it is plausible that requiring two of the strands not to hit each other has the effect of adding some fractional number of strands $f(q)$ between the strands required not to hit each other; similar phenomena have been observed elsewhere. When two strands meet at a point that happens to be on the external perimeter, the right side of one strand does not hit the left side of the other strand. Thus, κ_2' for the perimeter is κ_2 for the external perimeter, which (by $\kappa_2=\kappa_1/4=1/g$ [1] and $g_{\text{ext. perimeter}} = 1/g_{\text{perimeter}}$ [14]) in turn is $1/\kappa_2$ (for the perimeter), giving

$$\kappa_1/(2+f(q))^2=4/\kappa_1,$$

$$f(q)=\kappa_1/2-2,$$

$$\kappa_k'=\kappa_1/(k+\kappa_1/2-2)^2.$$

For example, when $q=0$ this predicts $\kappa_4'=2/9$. Indeed, the largest pinch point for SLE_8 corresponds to a triple point of the spanning tree, for which we already have the value 2/9. For other values of q, our measured values of κ_4' appear to be consistent with this formula.

More generally, when k strands meet at a point, and j adjacent pairs do not hit each other, we expect the winding angle variance to grow like

$$\frac{\kappa_1}{[k+j\max(0,\kappa_1/2-2)]^2}\ln L. \tag{1}$$

To measure the κ's for a given value of q, for each of several system sizes (side length L a power of two multiple of 4, 5, 6, or 7, starting with $L=4\times 2^3, 5\times 2^3, 6\times 2^3, 7\times 2^3, 4\times 2^4, \ldots$), we generated 10 000 random FK configurations using the methods described in Refs. [41,42]. In each one we identified the longest contour (and also the longest outer contour) and computed the winding angle function as defined above. For κ_2, the square of the winding at a random single edge on the loop is an estimator of the winding angle variance of the loop, but a more efficient estimator is the average square of the winding angle of edges on the loop. For κ_4', we measured the square of the winding at the pinch point of the longest pinch of the longest contour. The data for the longest contour when $q=1$ (percolation) are shown in Fig. 2, whose caption explains how we estimated κ_2, κ_2', and κ_4'. Tables II and III summarize our estimates.

We also conducted measurements for the minimum spanning tree (MST) with random edge weights. The paths of the MST are smoother and less windy than those of the UST (see Table III). For MST it is unlikely that $D_f=1+\kappa_1/8$, so the MST path is not described by SLE.

In conclusion, Eq. (1), which generalizes Duplantier's winding angle formula, is supported by both experiments and heuristic arguments. It would be interesting to see if Eq. (1) holds for SLE_κ.

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